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ON THE AXISYMMETRIC SCREW MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID*

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Equations of the axisymmetric screw motions (SM) of a viscous incompressible fluid are obtained. Exact solutions are found for the case of uniform SM. The criterion of the closeness of the SM to the uniform SM is formulated and exact non-linear solutions for the viscous SM with weak vorticity are derived.

The equations of the SM of a viscous incompressible fluid, i.e. of the motions in which the velocity and vorticity are collinear, were first studied by Steklov /1/ and then given in the most general form by Byushgens /2/. However, the examples of the solutions of this non-linear indeterminate system of equations are practically exhausted by the solutions for two special linear cases: one of Steklov /1/ ("uniform" SM, where the ratio of the velocity and vorticity moduli is constant), and the other of Caldonazzo /3/ (a uniform cylindrical SM). Only a few other examples of the use of Steklov-type solutions for some special type flows can be shown (see e.g. /4/ and the references there).

The indeterminacy of the system of viscous SM equations is its special feature. When the possibility of the existence of the spatial stationary SM of a viscous incompressible fluid described by a non-linear system of six equations in two unknowns was discussed in /2/, its compatibility was estimated in the general case to be of "low probability".

The use of any physical concepts in the analysis of the general case is largely hindered by the lack of demonstrable examples of real spatial flows which could be identified with SM. A unique non-trivial exception to this would appear to be the case with axial or spherical symmetry. This class could justifiably, in a sense, include such objects as a vortex behind a screw propeller, vortex tubes leaving wing tips, secondary flows in curved channels, a flow in a funnel, a tornado, swirling streams, etc. Moreover, one can speak of the possibility of creating such flows experimentally. This relatively simple special case retains, nevertheless, such essential characteristic features of SM as the indeterminacy of its description and the non-linearity, and this makes it possible to regard it, in a sense, as a model. The present paper deals with the analysis of an axisymmetric SM of a viscous incompressible fluid.

1. When describing the motion of the fluid we find that in the present case (the presence of an external potential force does not affect the generality of the approach) it is convenient to start off with the equations of motion of a viscous incompressible fluid in the Gromeka-Lamb form with respect to the vorticity $\boldsymbol{\omega} = (\omega_r, \omega_\varphi, \omega_z)$, written in a cylindrical system of coordinates. When this form is used, then taking into account the conditions of screw flow,

$$\Omega_\varphi V_z = \Omega_z V_\varphi, \quad \Omega_z V_r = \Omega_r V_z, \quad \Omega_r V_\varphi = \Omega_\varphi V_r \tag{1.1}$$

we obtain, without any changes, the system

$$\Omega_r^{\cdot} = \nu E^2 \Omega_r, \quad \Omega_\varphi^{\cdot} = \nu E^2 \Omega_\varphi, \quad \Omega_z^{\cdot} = \nu (E^2 + r^{-2}) \Omega_z \tag{1.2}$$
$$E^2 = \partial^2/\partial r^2 - r^{-1}\partial/\partial r + \partial^2/\partial z^2, \ \boldsymbol{\Omega} = r\boldsymbol{\omega}, \ \mathbf{V} = r\mathbf{v}$$

Moreover, the following equation of continuity must hold:

$$\partial V_r/\partial r + \partial V_z/\partial z = 0 \tag{1.3}$$

The relations connecting the velocity and vorticity components lead to relations of the form

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$$\Omega_r = -\partial V_\varphi/\partial z,\ \Omega_z = \partial V_\varphi/\partial r \qquad (1.4)$$
$$\Omega_\varphi = \partial V_r/\partial z - \partial V_z/\partial r + r^{-1}V_z$$

We see that according to (1.3) a stream function ψ exists such, that

$$V_r = \partial\psi/\partial z,\ V_z = -\partial\psi/\partial r$$

Substituting these expressions into the second relation of (1.1) and taking the first two relations of (1.4) into account, we conclude that

$$V_\varphi = V_\varphi(\psi) \equiv f(\psi) \qquad (1.5)$$

The first and third condition of (1.1) do not contradict the second condition, and also lead to the result (1.5).

Substituting the first relation of (1.4) into the first relation of (1.1) and taking into account (1.5), we obtain

$$\Omega_\varphi = -f df/d\psi$$

Comparing this equation with the result of substituting (1.5) into the last relation of (1.4), we obtain the following equation for ψ:

$$E^2\psi = F(\psi),\ F(\psi) \equiv -f df/d\psi \qquad (1.6)$$

The quantities $\Omega_r, \Omega_\varphi, \Omega_z$ expressed in terms of the stream function must satisfy the system of equations of dynamics (1.2). Appropriate substitutions yield

$$\partial Lf/\partial z = 0,\ \partial Lf/\partial r = 0 \qquad (1.7)$$
$$LF = 0,\ L = \partial/\partial t - \nu E^2$$

The first equation of (1.7) yields $Lf = a(t)$ where $a(t)$ is an arbitrary function of time which can be made equal to zero, since $a(t) \neq 0$ corresponds to the presence of an arbitrary irrotational field. Thus we have

$$Lf = 0 \qquad (1.8)$$

Comparing this equation with the last equation of (1.7) we see that, except for the trivial case of $f = F = 0$, the equations are compatible if and only if the functions f and F are connected by a linear relation, i.e.

$$-f df/d\psi = cf + c_0,\ c,\ c_0 = \text{const} \qquad (1.9)$$

The equation obtained represents a condition of compatibility of the overdefined system (1.6), (1.8) and has the following solution (apart from a constant):

$$\psi = -c^{-1}f + c_0c^{-2}\ln(f + c_0c^{-1}) \qquad (1.10)$$

Using (1.10) to pass, in Eq.(1.6), from ψ to f, we can obtain the second equation for f

$$(f + c_0c^{-1})[fE^2f + c^2(f + c_0c^{-1})] + c_0c^{-1}(f_r^2 + f_z^2) = 0 \qquad (1.11)$$

which forms, together with the linear parabolic Eq.(1.8), an overdefined system describing the axisymmetric SM of a viscous incompressible fluid. We note that the solution of the system of the form $f = -c_0c^{-1}$ is trivial, since it corresponds to irrotational flow.

2. Let us first consider two limiting cases of relation (1.9): $c_0 = 0,\ c = 0$.

In the first case the system of equations has the form

$$f_t = \nu E^2 f,\ E^2 f + c^2 f = 0 \qquad (2.1)$$

and has a non-stationary solution (the stationary solution is trivial)

$$f = \exp(-\nu c^2 t) G(r, z) \qquad (2.2)$$

where G is the solution of the equation $E^2G + c^2G = 0$. The solution is identical with that obtained by Steklov, but unlike in /1/, where the form of solution was found using only an equation of the type of the first equation of (2.1), its derivation is rigorous.

The last equation has, for example, a solution bounded at infinity, which generalizes the solution /3/ to the two-dimensional case

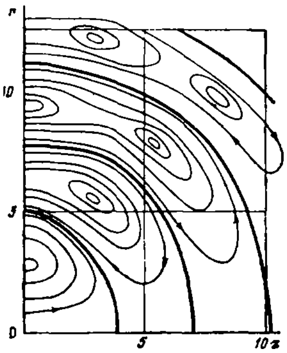
$$G = \exp(i\lambda z) r J_1(\sqrt{c^2 - \lambda^2}\, r),\ \lambda = \text{const}$$

where J_1 is a Bessel function. Using this solution we can construct, by virtue of the linearity of the system, a "fundamental" solution of system (2.1) independent of the wave number λ, say, in the following manner:

$$f^\circ(t, r, z) = \int_{-c}^{c} f(t, r, z, \lambda)\, d\lambda = -\pi \exp(-\nu c^2 t)\, \eta^{-1}(\xi^2 - \eta^2) \times$$

$$[J_0(\xi-\eta)J_1(\xi+\eta)+J_1(\xi-\eta)J_0(\xi+\eta)]$$
$$\zeta=\frac{1}{2}c\sqrt{r^2+z^2},\quad \eta=\frac{1}{2}cz$$

The stream lines of such a flow $f = \text{const}\ (\psi = -f/c)$ for a fixed instant of time are shown in the figure for $c = 1$ (the stream lines $\psi = 0$ are separated). We see that the flow has an axisymmetric lamellar structure with singularities resembling toroidal vortices.



A similar flow pattern was obtained in the course of numerical computation for the non-stationary problem of the motion of fluid between concentric spheres when the outer sphere was accelerated instantaneously /5/. We also note a certain resemblance between the pattern obtained and the cellular system of the secondary Taylor vortices in a flow between rotating cylinders, as well as the Hörtler vortices, and the qualitative similarity of their kinematics to that of the strict screw flows makes it possible to assume that such a similarity is definitely justified.

In order to analyse the second case $(c = 0)$, it is convenient to return from f to ψ:

$$E^2\psi = c_0,\quad \psi\psi_t - \nu c_0\psi + {}^1/_2\nu(\psi_r^2 + \psi_z^2) = 0 \qquad \textbf{(2.3)}$$

The problem of the compatibility of system (2.3) cannot be discussed without involving possible boundary value problems. In particular, when the fluid moves in a space not containing any boundaries, system (2.3) is found to be incompatible.

Indeed, the presence of the variables

$$t_0 = c_0 t,\ r_0 = \sqrt{c_0}r,\ z_0 = \sqrt{c_0}z$$

(the scale related to viscosity is not characteristic for the system, since the first equation of (2.3) is inviscid), in which the system of equations does not contain c_0 explicitly, means that the second equation of (2.3) must have a selfsimilar solution of the form

$$\psi = At_0^{\alpha}G(\zeta),\ \zeta = B(r_0^2 + z_0^2)/t_0$$

where A, B are arbitrary constants, $\alpha = 1$ and $G(\zeta)$ satisfy the equation

$$2\nu B\zeta G'^2 - \zeta GG' + G^2 - \nu G/A = 0 \qquad (2.4)$$

At the same time, the formal solution of such a class has, for the first equation of (2.3), the form

$$G(\zeta) = \zeta/(2AB) + \text{const}\cdot\zeta^{1/2}$$

which does not satisfy (2.4) and corresponds, in fact, to the solution of a more general form

$$\psi = {}^1/_2(r_0^2 + z_0^2) + T(t_0)R(r_0, z_0)$$

where $E^2R = 0$ and $T(t_0)$ is an arbitrary function not satisfying the second equation of (2.3) for any values of $T(t_0)$.

3. The last example shows that the problem of the compatibility of the general system (1.8), (1.11), i.e. of existence of solutions describing the axisymmetric viscous screw flows is, generally speaking, not trivial. The first example of Sect.2 shows that a definite class of solutions of system (1.8), (1.11) exists when the flow is homogeneous $(c_0 = 0)$. It would be natural to expect that the flows corresponding to sufficiently small values of c_0 (or of other quantities related to it), which we can assume to be close to the homogeneous screw flows, can be realized, and that solutions of system (1.8), (1.11) belonging to such class exist.

If we restrict our discussion to the class of flows including, in a continuous manner, such weakly inhomogeneous flows, then, before anything else, we must construct a criterion of inhomogeneity which could be used as the basis for formulating at least the necessary conditions for the flow to be close to homogeneous. With this purpose in mind, we shall consider the problem which was not considered until now, of choosing the arbitrary constants c and c_0 in (1.9). The constant, whose dimensions are of inverse length and velocity respectively, should carry the functions of the parameters determining the class of the flow, since they can be connected, in a natural manner, with the parameters characteristic for one or another class of flows. Thus, for example, the solution of the Steklov problem of the form (2.2) shows that in absence of the characteristic time scale and of the geometrical scale, their part is taken by $1/\nu c^2$ and $1/c$ respectively.

In the case of a non-stationary problem described by the system (1.8), (1.11) in terms of f, we will introduce the geometrical scale of the flow l, the characteristic time τ and the azimuthal velocity U. Using the dimensionless variables

$$\bar{t} = \frac{t}{\tau},\quad \bar{r} = \frac{r}{l},\quad \bar{z} = \frac{z}{l},\quad \bar{f} = \frac{f}{lU},\quad \bar{\psi} = \frac{\psi}{l^2U}$$

we obtain the system (1.8), (1.11) and relation (1.10) in the form (the bars are omitted)

$$\begin{gathered} f_t = \mathrm{Re}^{-1}E^2f \\ (f+\varepsilon)[fE^2f + \beta^2(f+\varepsilon)] + \varepsilon(f_r^2 + f_z^2) = 0 \\ \beta\psi = -f + \varepsilon\ln(f+\varepsilon) \\ \mathrm{Re} = \frac{l^2}{\nu\tau},\quad \beta = cl,\quad \varepsilon = \frac{1}{lU}\frac{c_0}{c} \end{gathered} \tag{3.1}$$

The system (3.1) must satisfy, for the class of flows including uniform flows, the demand of limiting passage, as $\varepsilon \to 0$, to the Steklov system for a homogeneous screw flow. This predetermines the independence of U from τ and l, which must therefore refer not to the azimuthal, but to the axial motion. Moreover, in accordance with the last equation of (3.1), we must choose $\beta = 1$, i.e. $c = l^{-1}$, and c_0 must be satisfied uniquely $c_0 = \nu l^{-1}$ in order to ensure the feasibility of $\varepsilon \to 0$. Here we have $\varepsilon = \nu/lU = \mathrm{Re}_\varphi^{-1}$ where Re_φ is the Reynolds number found from the characteristic azimuthal velocity.

The system obtained satisfies, for fixed Re, the demand of passage to the limit as $\varepsilon \to 0$ to the Steklov system, therefore the number Re_φ can serve as the criterion of inhomogeneity of the screw flow, and the nearness of ε to zero can characterize the degree of nearness of the flow to uniform. It can be shown that the system obtained has no stationary solutions. Indeed, if a non-trivial solution of the system

$$E^2\varphi = 0,\quad \varphi_r^2 + \varphi_z^2 = -\varepsilon^{-1}\varphi^2,\quad \varphi = f + \varepsilon \tag{3.2}$$

existed, it would in accordance with the form of the second equation, have to satisfy the condition $\varphi < 0$. Let us introduce the function Φ such, that $\varphi = -\Phi^{-2} < 0$, for which the system has the form

$$\Phi E^2\Phi = {}^3/_4\varepsilon^{-1},\quad \Phi_r^2 + \Phi_z^2 = {}^1/_4\varepsilon^{-1}$$

Planes or circular conic surfaces which are not solutions of the first of the above equations, correspond to the integrals of the second equation.

4. Below we shall give several examples to illustrate the fundamental possibility of the existence of solutions of the non-linear system (3.1). The examples will refer to a non-linear case of weakly vortical screw flows, when we will have (in dimensionless form)

$$\omega_i/v_i \ll 1,\quad i = r,\ \varphi,\ z$$

which is equivalent to the condition $\varphi \ll 1$ imposed on φ from (3.2).

In this case the non-degenerate system of lowest order obtained from (3.1) by neglecting terms of order $O(\varphi^3)$, has the form

$$\varphi_t = \mathrm{Re}^{-1}E^2\varphi,\quad \varphi E^2\varphi = \varphi_r^2 + \varphi_z^2 \tag{4.1}$$

In the case of a cylindrical screw flow $\varphi = \varphi(t, r)$ the system has an exact solution

$$\varphi(t, r) = \mathrm{const}\cdot\exp[-r^2/(4\varepsilon_0 t)],\quad \varepsilon_0 = \mathrm{Re}^{-1}$$

and the constant should be made, by virtue of the symmetry, equal to ε:

$$v_\varphi = r^{-1}f = \varepsilon r^{-1}\{\exp[-r^2/(4\varepsilon_0 t)] - 1\}$$

The solution is identical, with respect to the azimuthal velocity, with the classical solution describing the diffusion of a rectilinear vortex tube in a viscous fluid, but it has also an axial flow of velocity $v_z = -(2\varepsilon_0 t)^{-1} r v_\varphi$.

The presence of a region of retarded flow near the axis corresponds to the pattern observed experimentally in converging flows /6/.

In the general case $\varphi = \varphi(t, r, z)$ the system (4.1) has an exact solution

$$\varphi(t, r, z) = k\varepsilon\exp[\varepsilon_0\lambda^2 t - \lambda z - r^2/(4\varepsilon_0 t)],\quad \lambda = \mathrm{const} > 0$$

which grows without limit, beginning from some instant of time, at a fixed point of space. Such a type of solution can be correlated qualitatively with the effect of "collapse" of a rectilinear vortex /7/.

As was shown in Sect.3, system (3.1) as well as (4.1), has no strictly stationary solutions. However, when the **axial** Reynolds numbers Re are sufficiently large, certain quasistationary solutions satisfying system (4.1) with an accuracy of $O(\varepsilon_0)$ can be shown to exist.

Since the second equation of (4.1) is independent of ε_0, such a quasistationary approximation will be described by the system

$$\varphi_t = 0,\quad \varphi E^2\varphi = \varphi_r^2 + \varphi_z^2 \tag{4.2}$$

Making the substitution $\varphi = \exp(\theta)$ we reduce the non-linear equation to the linear form $E^2\theta = 0$, which has the following exact solutions bounded at infinity:

$$\theta = \begin{cases} -\alpha r^2 - \beta z, & \alpha, \beta = \text{const} > 0 \\ r \sum_{n=1}^{\infty} Z_1(\mu_n r) \exp(\mu_n z) \end{cases}$$

where Z_1 is a Bessel function of first order and first or second kind depending on the type of the complex quantity μ_n. Here the solution φ has a fairly complex form

$$\varphi = \text{const} \begin{cases} \exp(-\alpha r^2 - \beta z) \\ \exp\left[r \sum_{n=1}^{\infty} Z_1(\mu_n r) \exp(\mu_n z)\right] \end{cases}$$

The one-dimensional case of a cylindrical screw flow described by the exact solution of system (4.2)

$$\varphi = \text{const} \cdot \exp(-\alpha r^2), \quad \alpha = \text{const} > 0$$

which corresponds to the velocity field

$$v_\varphi = \varepsilon r^{-1} [1 - \exp(-\alpha r^2)], \quad v_z = 2\alpha r v_\varphi$$

lends itself to clearer interpretation. The solution is identical with the known solution for a time-limited state of a rectilinear vortex stretching in axial flow (Burgers vortex). The solution was used by a number of authors as a heuristic model of a twisted external flow, while studying the mechanism of vortex collapse /7/. The magnitude of the constant α can be estimated from the condition $v_\varphi, v_z = O(1): \alpha \sim \varepsilon^{-1}$ corresponding to the experimental results obtained by Garg /7/.

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ON STRONG TRANSITIONS BETWEEN STRUCTURES OF DIFFERING SYMMETRY ACCOMPANYING WEAKLY SUPERCRITICAL CONVECTION*

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A complete classification of the phase space of the dynamical system which describes the motion of a liquid when there is weakly supercritical convection is carried out within the framework of a six-mode Galerkin approximation. It is shown that all the phase trajectories are attracted to the corresponding stationary states. The domains of attraction to each of these states are found. The minimum value of a perturbation, which converts a weakly stable solution of one symmetry into a stable solution of another symmetry when the parameters of the problem are close to their bifurcation values, is estimated.

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